

PART III. LOCATIVE ONTO-LOGIC

10. PREORDERS AND LOCATIVE STRUCTURES

Generalia

10.1 We are going to provide a rather detailed comparison of locative structures with preorders. As the reader will see, locative structures generalize preorders.

The realm of locative structures is much broader in scope than the realm of preorders. Fortunately, locative structures enjoy a great wealth of the regularities of preorders. To make this clear, in what follows I am going to study rather carefully the counterparts of the basic features of preorders, which were brought together in chapters 4–6 above.

10.2 First of all, observe that each of the three types of location studied here is weaker than preordering.

This has already been established in (83) for locative structures fulfilling axioms from the second group: *If E is a preorder relation, then $\langle U, E \rangle$ satisfies both $L = P$ and $A = C$.*

Similarly for the remaining two families of axioms:

(101) *If $\langle U, E \rangle$ is a preorder relation, then $L = E = A$, i.e., $\langle U, E \rangle$ is locative.*

Proof. Assume that E is reflexive and transitive as well as the assumptions of $E \leq L$: for any x and y , xEy and zPx . By the last assumption: $\forall u (uEz \rightarrow uEx)$, hence $zEz \rightarrow zEx$. But E is reflexive, therefore zEx . Applying now transitivity of E , we obtain that zEy , as required.

The remaining inequality: $E \leq A$ can be checked analogously.

(102) *If E is a preorder relation, then $L = A$; i.e. $\langle U, E \rangle \models \mathbf{LA}$.*

This follows immediately by (83) and (101).

10.3 To resume:

- (103) *For preorders all five relations under consideration: E, P, C, L and A coincide, i.e., $L = P = E = C = A$.*

From the point of view of locative ontology preorders are therefore very regular, hence strong, structures.

Too regular, however. Recall that the distinction between P and C as well as between L and A discriminates two aspects of the parthood and the locative relations generated by the starting relation E . As we saw several times before, this distinction is important.

Yet it disappears in all preorders. Therefore, for a more discriminative study of location, preorders, a fortiori mereologies, are too powerful and too regular. In the realm of preorders we lose even the distinction between “to be a part of” and “to be located in”.

10.4 Anyway, preorders are rich in nice equations. In addition to those enumerated in (103) let me list here those equations studied previously which hold in the realm of preorders:

$P_E = E, P_{E^{-1}} = E^{-1}, (P_E)^{-1} = P_{E^{-1}}, L_E = E, L_{E^{-1}} = E^{-1}, (L_E)^{-1} = L_{E^{-1}}$ plus the analogous equations for $C_E, C_{E^{-1}}, A_E$ and $A_{E^{-1}}$.

Comparison

10.5 To what extent locative structures of the three kinds introduced previously in chapter 9 enrich our usual universe of discourse?

To answer this question we are going to provide a systematic comparison of the realms of the different locative structures with the realms studied previously: mereologies, premereologies, preorders and Leśniewski’s ontologies (cf. Figs. 5 and 6 in Ch. 7).

10.6 We accept the following convention concerning entailment: Conditions $A, B \vdash C, D$ or $A, B \not\vdash C, D$ mean respectively $A, B \vdash C$ and $A, B \vdash D$, $A, B \not\vdash C$ and $A, B \not\vdash D$; i.e. that A and B entails C and that it entails also D , and that A and B does not entail C and that it does not entail D .

Single cases $A \vdash B$ and $A \not\vdash B$ are obtained by a suitable particularization.

10.7 Let’s start with a consideration of the proper locative axioms: **INL**, **EXL** and **L**.

It is easy to see that each of them is weaker than the axioms of preorders (i.e. reflexivity *and* transitivity), as well as that the axioms of internal and external location are mutually independent, i.e., none of them entail the other:

(104) $\mathbf{PO} \vdash \mathbf{L}$ *and* $\mathbf{L} \not\vdash \mathbf{PO}$, $\mathbf{EXL} \not\vdash \mathbf{INL}$ *and* $\mathbf{INL} \not\vdash \mathbf{EXL}$.

As a matter of fact, the first three claims have been established already.

Indeed, by (101) we knew that $\mathbf{PO} \vdash \mathbf{L}$ and by Ex. 3 of §9.9 we learnt that $\mathbf{L} \not\vdash \mathbf{PO}$; in fact even more: $\mathbf{L} \not\vdash \mathbf{R}, \mathbf{T}$.

On the other hand, by Ex. 1 of §8.18 we know that $\mathbf{EXL}, \mathbf{T} \not\vdash \mathbf{INL}, \mathbf{R}$.

For the last claim consider the following

Ex. 6 Let $U = \{x, y, z\}$ and $E = \Delta \cup \{xy, yz, zy\}$. Its diagram is as follows:

$$\begin{array}{ccccc} \circ & \longrightarrow & \circ & \longleftarrow & \circ \\ x & & y & & z \end{array}$$

By a straightforward calculation we obtain that $P = \Delta \cup \{xy, zy\}$, $C = \Delta \cup \{zy, yz\}$, $L = \Delta \cup \{xy, yz, zy\}$ and $A = \Delta \cup \{yz, zy\}$. Thus $L = E$, $P < E$, $A = C$. Hence $\langle U, E \rangle \models \mathbf{INL}, \mathbf{AC}, \mathbf{PL}, \mathbf{R}$; whereas $\langle U, E \rangle$ falsifies $\mathbf{EXL}, \mathbf{L}, \mathbf{T}$.

A fortiori, $\mathbf{INL}, \mathbf{R} \not\vdash \mathbf{EXL}, \mathbf{T}$.

10.8 Notice that, by (98), using the converses of the models generated by examples Ex. 1 and Ex. 6 we obtain

(105) $\mathbf{INL}, \mathbf{T} \not\vdash \mathbf{EXL}, \mathbf{R}$ *and* $\mathbf{EXL}, \mathbf{R} \not\vdash \mathbf{INL}, \mathbf{T}$.

On the other hand observe that neither does location plus reflexivity imply transitivity nor does location plus transitivity imply reflexivity:

(106) $\mathbf{L}, \mathbf{R} \not\vdash \mathbf{T}$; $\mathbf{L}, \mathbf{T} \not\vdash \mathbf{R}$.

Proof. Let's start with the second claim: *location plus transitivity does not imply reflexivity*.

Consider the simplest case of a nonempty relation one can find:

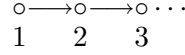
Ex. 7 $U := \{x, y\}$, $E := \{xy\}$, i.e.,

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ x & & y \end{array}$$

We immediately obtain that $P = \Delta \cup E$, $C = \Delta \cup E$ and $L = E = A$. Thus our model validates both transitivity (for $E \leq P$) and location but not reflexivity, as required.

To confirm the first claim: *location plus reflexivity does not imply transitivity*, consider the following model:

Ex. 8 $U := \{1, 2, 3, \dots\}$, $E := \{nn : n \geq 1\} \cup \{(n, n+1) : n \geq 1\}$. The diagram of this relation is quite familiar:



Let's calculate:

(i) $P = \Delta \cup \{(1, 2)\}$.

Take first the case $n = 1$. By definition of P , $1Pm$ iff $\forall i (iE1 \rightarrow iEm)$. But $iE1$ iff $i = 1$. Hence $1Pm$ iff $1Em$ iff $m = 1$ or $m = 2$, as needed.

Next, consider the case $n \geq 2$. nPm iff for any i , $iEn \rightarrow iEm$. But iEn iff $i = n - 1$ or $i = n$. Hence $n - 1Em$ and nEm . Therefore $m = n$. I.e., for $n \geq 2$, nPm iff $n = m$.

(ii) $C = \Delta$.

Let $n, m \geq 1$. By definition, nCm iff for any i , $mEi \rightarrow nEi$. But mEi iff $i = m$ or $i = m + 1$. Therefore, nCm iff nEm and $nEm + 1$ iff $n = m$.

(iii) $L = E$.

Let $n = 1$. $1Lm$ iff for any i , $iP1 \rightarrow iEm$. But $iP1$ iff $i = 1$. Hence $1Lm$ iff $1Em$, as required.

Consider now the case $n \geq 2$. By (i), iPn iff $i = n$. Therefore nLm iff nEm , again as required.

(iv) $A = E$.

Immediately by (ii), for if $C = \Delta$ then $A = E$.

10.9 As a matter of fact, in the last two sections the four conditions: reflexivity **R**, transitivity **T**, internal location **INL** and external location **EXL** have been compared: For brevity's sake rename the last two of them by **I** and **E**.

Connecting the above four relations in all *logically* different ways we obtain the following 15 formulas: **T**, **R**, **I**, **E**, **TI** (i.e. **T** \wedge **I**), **TE**, **TR** (**=PO**), **IE** (**=L**), **RI**, **RE**, **TRE**, **TIE**, **RIE** and **TRIE**. What are the logical connections between these relations, i.e., which implications connect them?

Joining together (101), (104), (105) and (106) by means of the standard logical laws, we see that the diagram given below describes all implications between the four conditions under consideration:

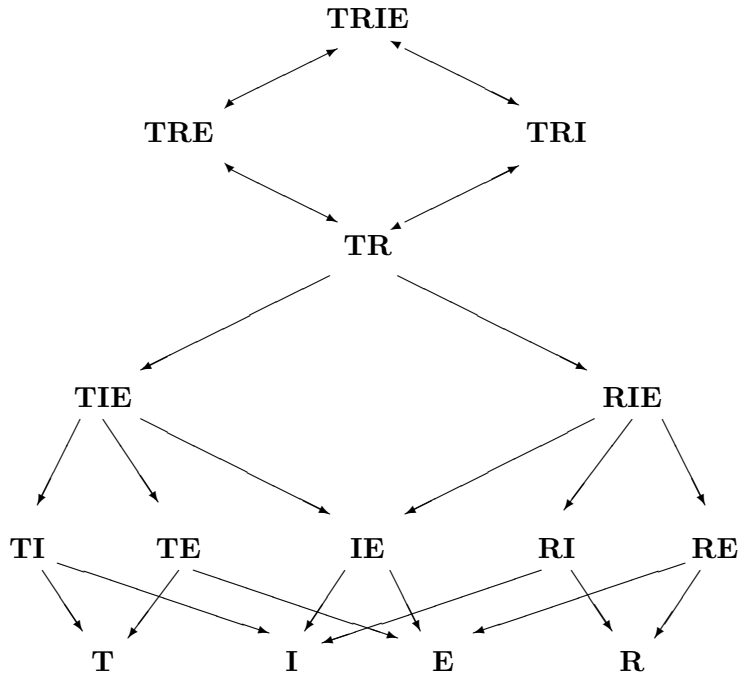


Fig. 8.

10.10 The model used in Ex. 8 calls into mind the realm of natural numbers. Let us discuss this matter more carefully.

The sequence of natural numbers is done by unlimited succession or infinite repetition. Usually, to describe this process the sequence function s is introduced: $s(n) := n + 1$. We start with 0. Next we put: $1 := s(0)$, $2 := s(1) = ss(0), \dots, n + 1 := s(n), \dots$

The model of generation of natural numbers is depicted by the following diagram:



As a matter of fact, \mathbf{N} models the function s . Its converse



describes the function s on the realm of negative integers, whereas their order-union:



gives account of the basic structure of all integers.

Observe that none of the above models is transitive or reflexive. A fortiori, they are not preorders. Notice, that their preorder closure models the relation \leq in each of the three respective domains.

Let me ask about the locative character of these basic arithmetical structures. By calculation similar to the one given in the second part of the proof of the last theorem we obtain

$$(107) \quad \mathbf{N} \models \mathbf{EXL} \text{ but } \mathbf{N} \not\models \mathbf{INL}, \underline{\mathbf{N}} \models \mathbf{INL} \text{ but } \underline{\mathbf{N}} \not\models \mathbf{EXL}, \text{ whereas } \\ \underline{\mathbf{N}} + \mathbf{N} \models \mathbf{L}.$$

In plain words: \mathbf{N} is externally, but not internally locative; $\underline{\mathbf{N}}$, conversely, is internally, but not externally locative, whereas the full structure of integers $\underline{\mathbf{N}} + \mathbf{N}$ is locative.

Thus, for a given n , the basic arithmetical function s indicates its location: n is *located in* $n + 1$.

Finally notice, that the above succession models emerge by infinitization of the Russian doll's model, investigated in §8.19 and claimed there to be the natural, paradigmatic, model for location.

10.11 Compare now locative ontology and Leśniewski's ontology. We know (cf. Chapter 7) that Leśniewski's axiom implies transitivity but it does not imply reflexivity. Therefore

$$(108) \quad \mathbf{L}, \mathbf{R} \not\models \mathbf{LON}: \text{location plus reflexivity doesn't imply Leśniewski's axiom.}$$

For otherwise $\mathbf{L}, \mathbf{R} \vdash \mathbf{T}$, which contradicts (106).

On the other hand,

$$(109) \quad \text{Leśniewski's axiom plus reflexivity does imply location: } \mathbf{LON}, \mathbf{R} \vdash \mathbf{L}.$$

For $\mathbf{LON} \wedge \mathbf{R}$ is stronger even than preorders \mathbf{PO} .

By Ex. 7 of §10.8 we also know that *location plus transitivity does not imply Leśniewski's axiom*:

$$(110) \quad \mathbf{L}, \mathbf{T} \not\models \mathbf{LON}$$

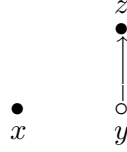
for in the model of Ex. 7 we have xEy but not xEy , contradicting one of the essential ingredients of Leśniewski's Ontology, cf. (42) of chapter 7.

Also, *Leśniewski's axiom does not imply location*:

(111) **LON** $\not\vdash$ **L**.

To see this consider the following

Ex. 9 $U := \{x, y, z\}$, $E := \{yy, yz\}$. Its diagram is as follows



Clearly this model verifies **LON** but falsifies **L**, for y is related to itself but not located in itself: yEy but $\neg yLy$.

Finally, ask *whether location plus Leśniewski's axiom imply reflexivity*? The answer again is in negative:

(112) **L**, **LON** $\not\vdash$ **R**

To see this it suffices to realize that the model containing only one simple item: $U = \{x\}$, $E = \emptyset$ is both locative and Leśniewskian, but not reflexive.

In conclusion, locative ontologies are new also in comparison with Leśniewski's structures.

10.12 In the last five sections quite a lot was calculated. A rather detailed description of the logical interconnections between the seven concepts: the *four* traditional - transitivity, reflexivity, preordering and Leśniewski's condition, and the *three* introduced here - internal location, external location and location has been presented.

For brevity's sake the notation introduced by the convention B is simplified here in the two basic cases: we write **IS** and **ES** instead of respectively **INLS** and **EXLS**, i.e., they respectively denote the class of all internally locative and the class of all externally locative structures.

The content of the last five sections can now be summarized by:

- (113) *i*) **TS** \cap **RS** = **POS** $\not\subseteq$ **LS** = **IS** \cap **ES**
ii) **LS** $\not\subseteq$ **IS**, **ES**
iii) **RS** \cap **LS** - **POS** $\neq \emptyset$, **TS** \cap **LS** - **POS** $\neq \emptyset$
iv) **LS** - **TS** $\neq \emptyset$, **TS** - **LS** $\neq \emptyset$, **LS** - **RS** $\neq \emptyset$ and **RS** - **LS** $\neq \emptyset$
v) **LONS** $\not\subseteq$ **TS**, **POS** \cap **LONS** $\not\subseteq$ **LS** \cap **LONS**

- vi) $\mathbf{IS} \cap \mathbf{RS} - \mathbf{LS} \cap \mathbf{RS} \neq \emptyset$, $\mathbf{ES} \cap \mathbf{RS} - \mathbf{LS} \cap \mathbf{RS} \neq \emptyset$;
 $\mathbf{IS} \cap \mathbf{RS} - \mathbf{ES} \neq \emptyset$, and $\mathbf{ES} \cap \mathbf{RS} - \mathbf{IS} \neq \emptyset$
- vii) $\mathbf{IS} \cap \mathbf{TS} - \mathbf{LS} \cap \mathbf{TS} \neq \emptyset$, $\mathbf{ES} \cap \mathbf{TS} - \mathbf{LS} \neq \emptyset$;
 $\mathbf{IS} \cap \mathbf{TS} - \mathbf{ES} \neq \emptyset$, and $\mathbf{ES} \cap \mathbf{TS} - \mathbf{IS} \neq \emptyset$
- viii) $\mathbf{RS} - \mathbf{IS} \cup \mathbf{ES} \neq \emptyset$, $\mathbf{TS} - \mathbf{IS} \cup \mathbf{ES} \neq \emptyset$; $\mathbf{LONS} - \mathbf{IS} \cup \mathbf{ES} \neq \emptyset$

The above interrelations are pictured in the following *Ontologische Salzburgerkugel*:

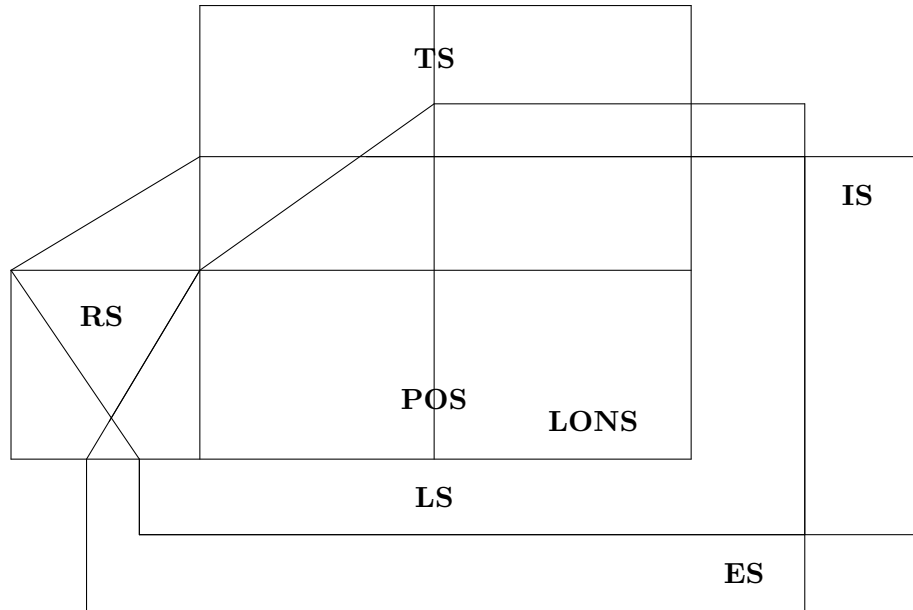


Fig. 9.

Comparison Continued

10.13 Turn now to domains satisfying axioms from the second family. By (81)–(87) from the sections 9.27–9.29 we know

- (114) i) $\mathbf{PLS} = \mathbf{RS} \cap \mathbf{CAS}$
 ii) $\mathbf{TS} \not\subseteq \mathbf{PLS} \cap \mathbf{CAS}$
 iii) $\mathbf{POS} \not\subseteq \mathbf{LPS} \not\subseteq \mathbf{RS}$ and $\mathbf{POS} \not\subseteq \mathbf{ACS} \not\subseteq \mathbf{RS}$

Therefore, locative domains of the *second* type, like the basic domains of the first type, extend preorders, but in not so deviant way, for usually they are reflexive.

On the other hand, by (113i) and (114iii), structures from the first family are, in a sense, complementary to ones from the second family. Taken separately they extend preorders in two different directions, while taken together they approximate them:

- (115) *i)* $\mathbf{LPS} \cap \mathbf{IS} = \mathbf{POS} = \mathbf{ACS} \cap \mathbf{ES}$, *a fortiori*
- ii)* $\mathbf{LPS} \cap \mathbf{LS} = \mathbf{POS} = \mathbf{ACS} \cap \mathbf{LS}$.

Using the terminology of §9.11 we can say that preorders are exactly the locative and prelocative, or the locative and preallocative, orders.

10.14 Notice that

- (116) \mathbf{LPS} and \mathbf{ACS} intersect each other.

To see this cf. Ex. 6 of §10.7 and its converse.

- (117) \mathbf{LPS} and \mathbf{LS} , as well as \mathbf{ACS} and \mathbf{LS} , intersect each other.

Indeed, by (115), both pairs have the same intersection, namely the family of all preorders \mathbf{POS} .

To see that \mathbf{L} does not imply either \mathbf{LP} or \mathbf{AC} use the infinite succession model of §10.10; whereas Ex. 6, with its converse, can be used to check that $A = C$, respectively $L = P$, does not imply $L = E = A$

10.15 The above interconnections, simplified by considering only one sort of basic locative structures (we omit \mathbf{IS} and \mathbf{ES}), are pictured in the second *Salzburgerkugel*:

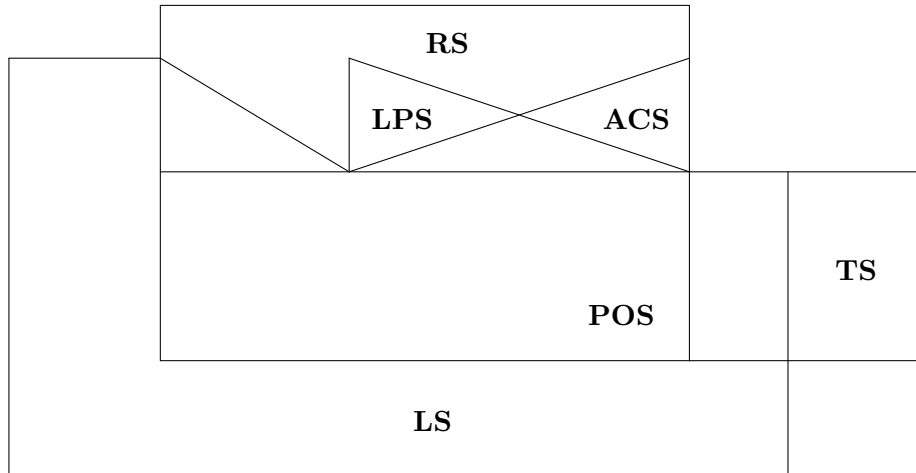


Fig. 10.

10.16 Finally, for the third group of locative structures, by the discussion of §9.32, we immediately obtain

$$(118) \quad \mathbf{ALS} \cap \mathbf{IS} \subsetneq \mathbf{ES} \text{ and } \mathbf{ALS} \cap \mathbf{IS} \subsetneq \mathbf{ES}. \text{ Also } \mathbf{LS} \subsetneq \mathbf{ALS}.$$

As a matter of fact

$$(119) \quad \mathbf{ALS} \cap \mathbf{IS} = \mathbf{LS} = \mathbf{ALS} \cap \mathbf{ES}$$

The above situation is depicted in the third *Salzburgerkugel*:

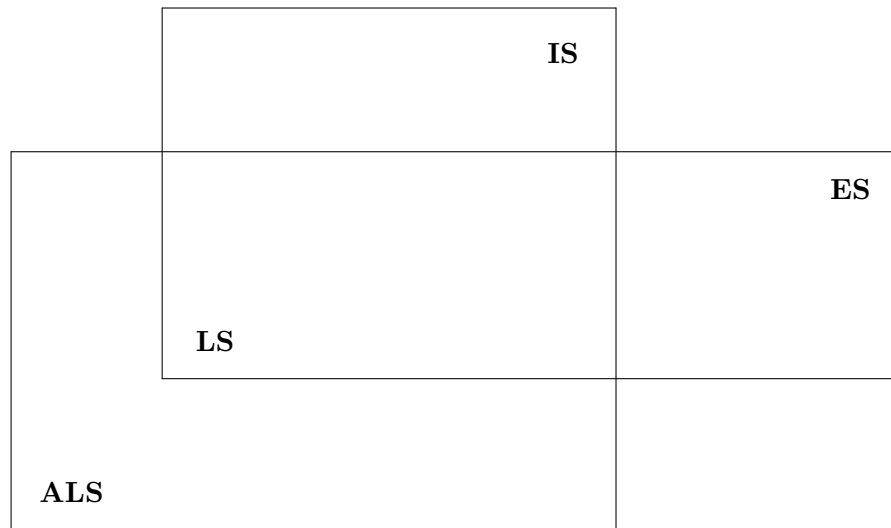


Fig. 11.

Mereolocation

10.17 In the previous sections of the present chapter our attention has been confined to comparison of preorders with locative structures of several types, i.e., to the investigation of the logical relations connecting axioms comparing the five relations involved: E , P , C , L and A . Three cases appear to be particularly important: preorders in and of themselves — **POS**, and the two basic families of locative structures — **IS** and **ES**, i.e., axioms: $E = P$, $E = L$ and $E = A$, respectively.

Notice that until now we did not compare the mereological relation M with location L . To complete our discussion I am going therefore to consider M once again.

10.18 Previously, in Chapter 6, M was compared with E : $E = M$, which defines premereologies. It was also compared there with P : $M = P$, which defines a bigger family of structures equal to premereologies in the realm of preorders, cf. (15) of §6.6.

Following our custom, **PS** and **MPS** are used to denote respectively the family of all premereologies (cf. 6.33) and the family of all relations satisfying **MP**: $M = P$.

By (15) we have

$$(120) \quad \mathbf{PS} = \mathbf{POS} \cap \mathbf{MPS}$$

10.19 Observe that M , like L , O and P (cf. respectively (43), (23) and (2)), satisfies the following monotonicity law:

$$(121) \quad zPx \wedge xMy \rightarrow zMy.$$

Proof. Assume: i) zPx , i.e. $(z] \subseteq (x]$, ii) xMy , i.e. for any u , $uOx \rightarrow uOy$, and iii) uOz , i.e. for some w , $wEu \wedge wEz$. By i), wEx . Hence uOx . Thus, by ii), uOy . Therefore $uOz \rightarrow uOy$, i.e. zMy , as required.

10.20 A metalogical conclusion is in order. The monotonicity of a given relation with respect to the parthood relation is too general, hence logically too weak, to characterize a given type of relation, for it fails to distinguish between P , M , O and L .

10.21 The relation E is said to be *mereolocative* iff its conjugate mereological and locative relations coincide:

$$\mathbf{ML} \qquad M = L.$$

The family of all mereolocative structures is denoted by **MLS**. Clearly, mereological containment is mereolocative:

$$(122) \quad \mathbf{ME} \text{ implies } \mathbf{ML}, \text{ i.e., } \mathbf{PS} \subseteq \mathbf{MLS}.$$

Indeed, premereologies are preorders, which, in turn, are locative structures. Hence, $E = M$ and $E = L$, therefore $M = L$.

10.22 Is the reverse implication true? Does mereolocation coincide with mereological containment?

In what follows we will answer this question in the negative, thus clarifying that mereolocation is an essential weakening of the mereological relation.

It does, however, preserve some important features both of location and of the mereological relation.

10.23 Notice that

(123) *If E is mereolocateive, then L is a preorder relation.*

Indeed, by the assumption L equals M which, by (12), is known to be a preorder relation.

(124) *If E is mereolocateive, then E is reflexive, but not conversely:*
MLS \subsetneq **RS**.

Proof. By the previous claim, L is reflexive, which, by (50), is equivalent to the reflexivity of E . To see that the inclusion is proper take, for example, Ex. 7 of §10.8.

10.24 As a matter of fact, the axiom **ML** introduces a new class of structures which intersects each of the following three classes of structures: preorders, transitive structures and internally locative structures with the same trace — the class of all premereologies.

This fact is stated in the following sequence of claims:

(125) *If E is transitive and mereolocateive, then E is internally locative:*
TS \cap **MLS** \subseteq **IS**.

Proof. By the first assumption and (18) we have $E \leq M$, whereas by the second one: $M = L$, hence $E \leq L$, which — by (69) — is equivalent to **INL**.

(126) *Premereologies are exactly those relational structures which are both premereological and internally locative: **PS** = **IS** \cap **MLS**.*

Proof. By (13), (102) and (122) the right-hand inclusion is immediate.

For the reverse inclusion assume that $L = E$ and $M = L$. Hence $M = E$, as required.

Therefore, the axiom **ML** restricts the family of all internally locative structures to premereologies, as **MP** does in the realm of preorders.

(127) *Premereologies are exactly those relational structures which are both premereological and transitive: **PS** = **TS** \cap **MLS**.*

Proof. The right-hand inclusion follows again from (13) and (124).

For the left-hand inclusion apply (125): $\mathbf{TS} \cap \mathbf{MLS} \subseteq \mathbf{IS}$. Hence $\mathbf{TS} \cap \mathbf{MLS} \subseteq \mathbf{IS} \cap \mathbf{MLS}$. Using now (126) we obtain the required inclusion: $\mathbf{TS} \cap \mathbf{MLS} \subseteq \mathbf{PS}$.

Analogously

$$(128) \quad \mathbf{PS} = \mathbf{MLS} \cap \mathbf{POS}.$$

Observe that

(129) \mathbf{IS} , \mathbf{TS} and \mathbf{MLS} cross each other. Also \mathbf{MLS} and \mathbf{POS} cross each other.

Proof. To prove the above claims we need a number of models. Consider first the following

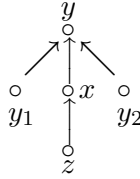
Ex. 10 $U := \{i : i \geq 1\}$ $E := \{ji : i, j \geq 1 \text{ and } j = i \text{ or } j = i + 1\}$

$$\begin{array}{ccccccc} \dots & \circ & \longrightarrow & \circ & \longrightarrow & \circ & \\ & & & 3 & & 2 & & 1 \end{array}$$

E is reflexive, i.e. $\Delta \subseteq E$, but it is not transitive. We can easily check that $M = \Delta = P$, and $E = L$. Hence $M \not\subseteq L$. In conclusion: $\mathbf{MP}, \mathbf{INL} \not\vdash \mathbf{ML}, \mathbf{T}$; hence both \mathbf{MPS} and \mathbf{IS} are included neither in \mathbf{MLS} nor in \mathbf{TS} .

Next, take

Ex. 11 $U := \{z, x, y, y_1, y_2\}$ $E := \Delta \cup \{zx, xy, y_1y, y_2y\}$



Here we can calculate that $P = \Delta \cup \{zx, y_1y, y_2y\} = L = M$. Hence $L < E$. E again is reflexive, but not transitive.

In conclusion: $P = M$, $L = M \not\vdash \mathbf{INL}, \mathbf{T}$; hence both \mathbf{MPS} and \mathbf{MLS} are included neither in \mathbf{IS} nor in \mathbf{TS} .

In the next turn, consider the simplest model of all: a singleton with the empty relation.

Ex. 12 $U := \{x\}$, $E := \emptyset$.

Here $P = \{xx\} = M$, whereas $L = \emptyset = E$. Moreover, E is transitive, but not reflexive. A fortiori, it is not a preorder relation.

Hence **T,MP** $\not\vdash$ **INL,ML,R,PO**. Inter alia, **TS** $\not\subseteq$ **IS** and **TS** $\not\subseteq$ **MLS**.

This finish the proof of the first claim.

To check that **MLS** and **POS** cross each other, notice firstly that **MLS** is not included in **POS**, because by Ex. 11 it, but not **POS**, is not included in **TS**; notice secondly that **POS** is not included in **MLS**, for otherwise **PS** = **MLS** \cap **IS** = **POS**, which contradicts §6.5. QED

The connections calculated in the present section can be summarized in the following *fourth ontological Salzburgerkugel*:

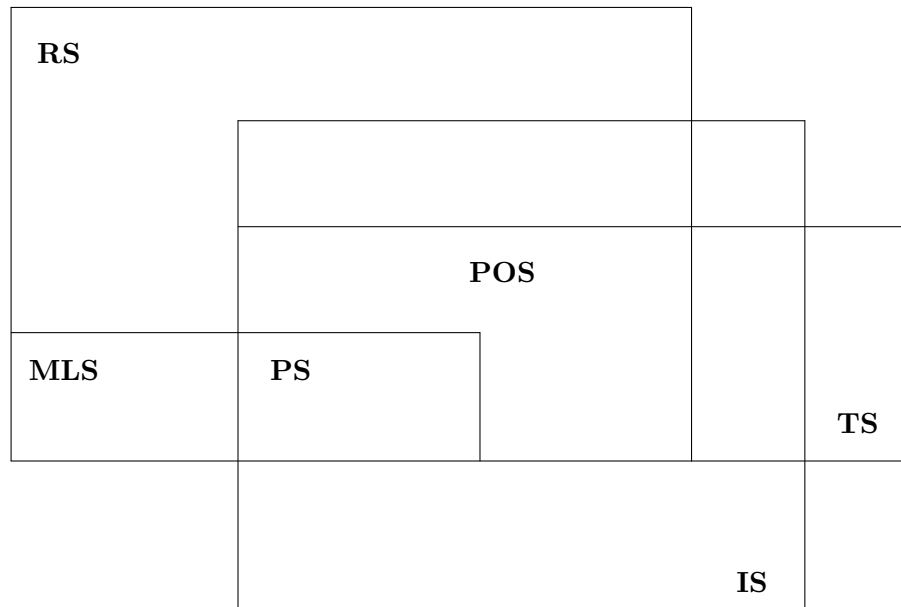


Fig. 12.

10.25 Turn now to the question of the interdependence between our twin-axioms:

ML $M = L$ and

MP $M = P$.

By the third model used in the proof of theorem (129), i.e. by Ex. 12, we know that

(130) **MP** does not imply **ML**, even more: **MP, T** $\not\vdash$ **ML**.

10.26 What is true of the reverse implication? Observe first that in general

(131) **ML** implies $P \leq M$.

Proof. Assume $L = M$ and $\neg(P \leq M)$. I.e., for some x and y , xPy and $\neg(xMy)$. Hence $\neg(xLy)$, i.e., for some u : uPx and (i) $\neg(uEy)$. By xPy , uPx and transitivity of P we have uPy . Therefore, if zEu then zEy . But by (124) E is reflexive, hence uEu . Therefore uEy , which contradicts (i).

ML seems and really is a rather strong axiom. It implies, inter alia, that L and E are reflexive, hence $P \leq L \leq E$. On the other hand, as we just proved, it implies $P \leq M$.

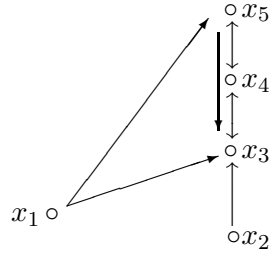
However, it is too weak to imply the reverse inequality: $M \leq P$.

(132) **ML** does not imply **MP**.

Proof. Consider the following model:

Ex. 13 $U := \{x_1, x_2, x_3, x_4, x_5\}$
 $E := \Delta \cup \{x_1x_3, x_1x_5, x_2x_3, x_3x_4, x_4x_3, x_4x_5, x_5x_4, x_5x_3\}$

Its diagram is as follows:



Now we can check that $P = \Delta \cup \{x_1x_3, x_1x_5, x_2x_3, x_4x_3\}$, $L = \Delta \cup \{x_1x_3, x_1x_5, x_2x_3, x_4x_3, x_4x_5, x_5x_3\} = M$. Hence $\Delta < P < L = M < E$.

Therefore Ex. 13 verifies **ML** but falsifies **MP**, a fortiori **MP**, as required.

10.27 The information concerning **MP** which we collected in (15), (120), (130)–(132) enables us to complete the map of Fig. 12 by the following *fifth Salzburgerkugel*:

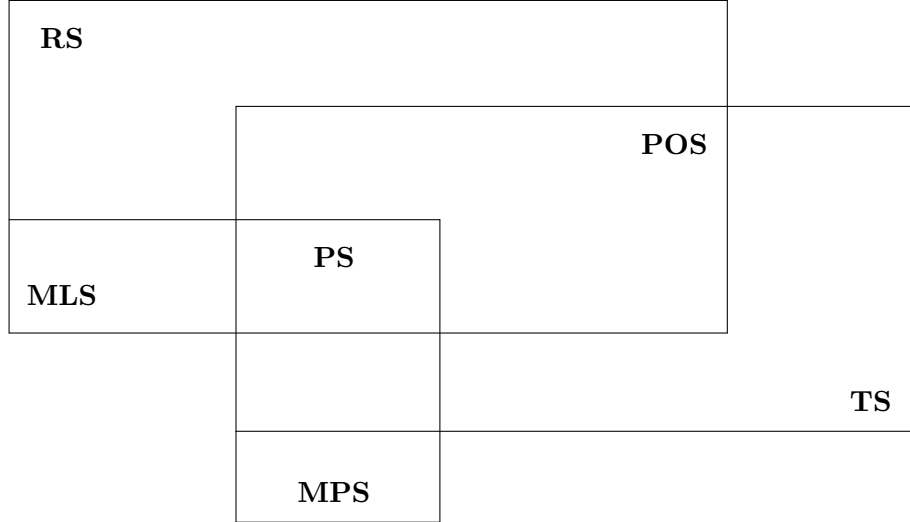


Fig. 13.

10.28 It is clear that the axiom of mereolocation implies the “condensing” part of the premereological axiom of condensation:

$$(133) \quad M = L \rightarrow M \leq E.$$

What we can say truly on the remaining part of **ME**? Call the suitable formula the axiom of *semitransitivity*:

$$**EM** \quad E \leq M.$$

The reason for this name is as follows: in the claim (18) of §6.7 it was stated that transitivity entails $E \leq M$.

On the other hand, it is easy to check that **EM** does not imply **T**. To this end take the following intransitive model:

$$\text{Ex. 14} \quad U := \{x, y\}, \quad E := \{xy, yx, xx\}$$

$$\begin{array}{ccc} \circ & \longleftrightarrow & \bullet \\ x & & y \end{array}$$

Here $P = \Delta \cup \{yx\}$, $L = \{yx\}$ and $O = U^2 = M$. Therefore $E \leq M$, but P crosses E , as required.

In conclusion

- (134) *Each transitive frame is semitransitive, but not conversely:*
 $\mathbf{TS} \not\subseteq \mathbf{EMS}$.

We immediately obtain

- (135) *Each semitransitive and mereolocative frame is internally locative:*
 $\mathbf{EMS} \cap \mathbf{MLS} \subseteq \mathbf{IS}$.

Clearly

- (136) *Each premereology is both semitransitive and mereolocative:*
 $\mathbf{PS} \subseteq \mathbf{EMS} \cap \mathbf{MLS}$.

Using (135), (136) and (126) we obtain the following generalization of (126):

- (137) *Premereologies are exactly these frames which are both semitransitive and mereolocative:* $\mathbf{PS} = \mathbf{EMS} \cap \mathbf{MLS}$.

Proof. We need only to check the left-hand inclusion, which, by (135), is immediate: $\mathbf{EMS} \cap \mathbf{MLS} \subseteq \mathbf{IS} \cap \mathbf{MLS} = \mathbf{PS}$.

10.29 Finally notice

- (138) *Semitransitive \mathbf{MP} -frames are transitive, but not conversely:*
 $\mathbf{EMS} \cap \mathbf{MPS} \not\subseteq \mathbf{TS}$.

Proof. Assume $E \leq M = P$. Hence $E \leq P$ which, by (4), is known to characterize transitivity.

To prove the second claim take any preorder which is not premereological, for example: each nontrivial Boolean algebra. Recall (15): In the realm of preorders \mathbf{MP} characterizes premereologies. Hence any Boolean algebra verifies \mathbf{T} but falsifies \mathbf{MP} . Therefore \mathbf{T} does not imply \mathbf{MP} , as was implicitly claimed.

10.30 And the last, but not least, move in the present search for reasonable modifications of a few well-established structures concerns the case of semimereolocation and the case of semiprelocation.

The relation E is said to be *semimereolocative* iff it fulfils the following equation:

$$\mathbf{SML} \qquad M \cap E = L;$$

whereas it is called *semiprelocative* iff the following equality holds:

$$\mathbf{SPL} \quad P \cap E = L.$$

First of all

(139) *If E is mereolocative, then E is semimereolocative, but not conversely: $\mathbf{MLS} \subsetneq \mathbf{SMLS}$.*

Proof. As regards the first claim assume $M = L$. We know that $L \leq E$, i.e., $L = L \cap E$. Hence $L = L \cap E = M \cap E$, as required.

For the second claim consider the following

$$\mathbf{Ex. 15} \quad U := \{x, y, z\}, \quad E := \{xy, yz\}$$

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ x & & y & & z \end{array}$$

Here $P = \Delta \cup \{xy\}$, $L = \{xy\}$ and $M = \Delta \cup \{xy, xz\}$. Thus $M \cap E = L$, but $M \neq L$, as required. Observe that in our model E , P and M cross each other.

(140) *If E is prelocative, then E is semiprelocative, but not conversely: $\mathbf{PLS} \subsetneq \mathbf{SPLS}$.*

Proof. The argument for the first claim is, *mutatis mutandis*, the same as the argument for the analogous claim in the previous theorem.

To prove the second claim return to Ex. 7 of §10.8. There indeed $P \cap E = L$, but $P \neq L$, as required.

10.31 To locate new structures with respects to preorders observe:

(141) *If E is a preorder (and thus, a fortiori, premereology), then E is semimereolocative and semiprelocative. None of the above inclusions is reversible.*

Proof. Assume that E is a preorder relation, hence $E = P = L$. By (134), $E = E \cap M$. Therefore $L = E \cap M$.

The second inclusion immediately follows from (140).

To prove that semimereolocation does not imply transitivity, hence the axiom **PO** of preorders, use again Ex. 15; whereas the last claim can be checked by use of Ex. 7, where $P \cap E = L$, but E is not reflexive, hence it falsifies **PO**.

10.32 Confront now both types of frames just introduced with suitable versions of transitivity:

- (142) *i) If E is semitransitive and semimereolocative, then E is internally locative, but not conversely: $\mathbf{EMS} \cap \mathbf{SMLS} \subsetneq \mathbf{IS}$*
ii) If E is transitive and semiprelocative, then it is internally locative, but not conversely: $\mathbf{TS} \cap \mathbf{SPLS} \subsetneq \mathbf{IS}$

Proof. Assume $E \leq M$ and $M \cap E = L$. Hence $E = M \cap E = L$, as required.

The first claim of ii) is proved by the same, *mutatis mutandis*, argument.

For the second claims of both i) and ii) reconsider the model $\mathbf{N}+\mathbf{N}$ of §10.10. There $E = L$, $E \cap \Delta = \emptyset$ and $M = \Delta = P$. Therefore $M \cap E = \emptyset = P \cap E$, but both L and P are nonempty.

Notice that (142i) generalizes (135).

- (143) *i) Internally locative and semimereolocative frames are semitransitive, but not conversely: $\mathbf{IS} \cap \mathbf{SMLS} \subsetneq \mathbf{EMS}$*
ii) Internally locative and semiprelocative frames are transitive, but not conversely: $\mathbf{IS} \cap \mathbf{SPLS} \subsetneq \mathbf{TS}$

Proof. Ad i) Assume $L = M$ and $M \cap E = L$. Then $M \cap E = E$, i.e., $E \leq M$, as required.

To check the second claim consider, for example, Ex. 14 of §10.28 where indeed $L < E < M$, hence \mathbf{ST} does not imply \mathbf{INL} .

As immediate corollaries we have

- (144) *i) Semimereolocative and semitransitive frames are exactly semimereolocative and internally locative frames:*

$$\mathbf{EMS} \cap \mathbf{SMLS} = \mathbf{IS} \cap \mathbf{SMLS}$$

- ii) Frames which are semiprelocative and transitive are exactly semiprelocative and internally locative ones:*

$$\mathbf{TS} \cap \mathbf{SPLS} = \mathbf{IS} \cap \mathbf{SPLS}$$

Proof. Ad i) By (142i) $\mathbf{EMS} \cap \mathbf{SMLS} \subseteq \mathbf{IS} \cap \mathbf{SMLS}$, whereas by (143i) $\mathbf{IS} \cap \mathbf{SMLS} \subseteq \mathbf{EMS} \cap \mathbf{SMLS}$, hence both intersections coincide: $\mathbf{IS} \cap \mathbf{SMLS} = \mathbf{EMS} \cap \mathbf{SMLS}$.

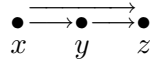
To prove the claim ii) use a similar combination of (142ii) and (143ii).

10.33 In conclusion: Transitivity and semitransitivity play quite similar, interchangeable, role in combination respectively with semiprelocation and semimereolocation, in both cases indicating internally locative structures.

As regards appropriate intersections, they clearly contain preorders: $\mathbf{POS} \subseteq \mathbf{TS} \cap \mathbf{SPLS}$, $\mathbf{EMS} \cap \mathbf{SMLS}$. Are these inclusions proper or not?

Yes, they are. As regards the second inclusion: $\mathbf{POS} \subsetneq \mathbf{EMS} \cap \mathbf{SMLS}$ use Ex. 12 of §10.24, whereas the properness of the first inclusion: $\mathbf{POS} \subsetneq \mathbf{TS} \cap \mathbf{SPLS}$ follows by checking the following

Ex. 16 $U := \{x, y, z\}$, $E := \{xy, xz, yz\}$



Here $L = E$ and $P = \Delta \cup E$, hence $E = L < P$, as needed.

10.34 The above connections concerning semimereolocation lead to the following *sixth Salzburgerkugel*, being a refined version of Fig. 12 (its dashed part corresponds to $\mathbf{IS} \cap \mathbf{SMLS} = \mathbf{IS} \cap \mathbf{EMS}$):

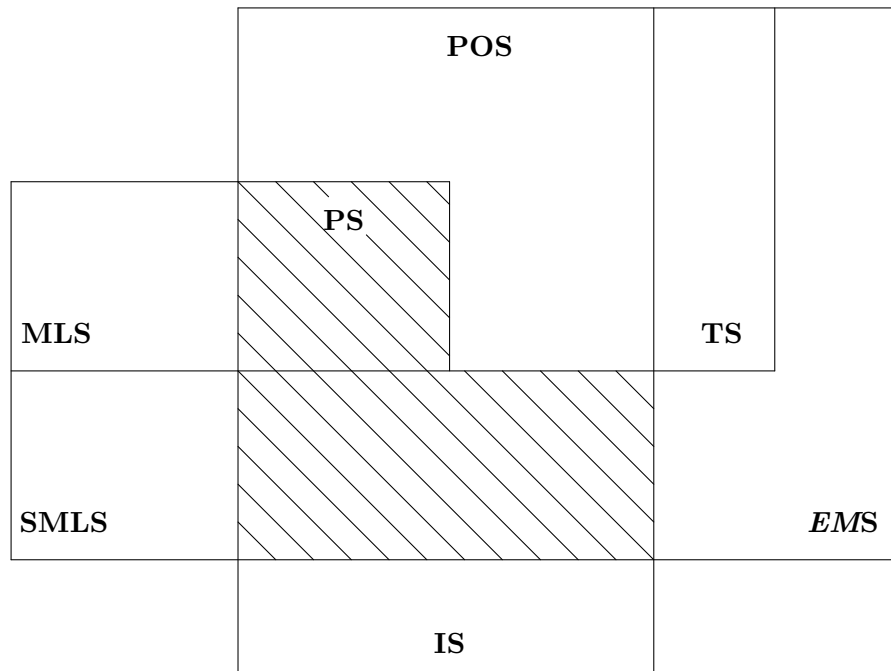


Fig. 14.

An analogous diagram can be drawn to indicate interrelations which characterize prelocations.

Conclusion

10.35 The complex picture of all connections pointed out in the present chapter can now be obtained by joining together all six Salzbergerkugeln. This is left to the reader's imagination, because the picture is too complex for the rather restricted drawing abilities of the present author.

10.36 Semitransitivity indeed seems to be a reasonable generalization of transitivity. This suggests a suitable generalization of the notion of preorder.

Let a reflexive and semitransitive relation E be called a *semipreorder* relation. The family of all semipreorders is denoted by **SPOS**. Clearly

(145) E is a semipreorder relation iff $P \leq E \leq M$.

The place of semipreorders with respect to other kinds of frames can easily be delineated by analysis of the previous diagrams. Inter alia

- (146) *i) Semimereolocative semipreorders are internally locative, but not conversely: $\mathbf{SPOS} \cap \mathbf{SMLS} \subsetneq \mathbf{IS}$;*
ii) Semiprelocative semipreorders are prelocative, but not conversely: $\mathbf{SPOS} \cap \mathbf{SPLS} \subsetneq \mathbf{PLS}$.

10.37 The variety of reasonable types of relational frames is both amazing and amusing. In this chapter we touch only a small, but important part of it. What we saw, however, calls for further research and attention.

(To be continued)

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