PART II. LOCATION

The most subtle points of formalisation are: first, fix intuition and, next, find adequate and as simple as possible formal means to express and develop it.

Our investigation of location starts with its informal discussion. After fixing intuition several clues concerning location are axiomatized in the relational framework and, next, developed in a very preliminary way.

8. BASIC CONCEPTS, KINDS AND PROPERTIES OF LOCATION

The Idea

8.1 The idea of location is both general and fundamental.

Location has several sides: *topological* — find a place, take it and fill or cover it; *geometrical* or *physical* — locate by fixing (space-time) coordinates, *practical* — locate successfully and economically, and so on.

Therefore, location has a wealth of connections to such items as measure, similarity, etc.

8.2 Location has also a quite remarkable *ontological* dimension.

It has been pointed out in chapter 2 that some basic, primitive *be*-statements are locative ones.

Consider, for example, a sentence: *I am here*, or its specification: *I am in Schaan*. This locates me in a nice city of Liechtenstein. But what does this exactly mean?

Clearly, I am a compound item. Hence, if I am in Schaan, then everything of which I am compounded must also be in Schaan. This includes my body with all its parts (if one of my legs is in Buchs, then I am not in Schaan, but between it and Buchs). But it also includes my minds' activities, including my standard behaviour, etc. I am there — physically, mentally, professionally, etc.

8.3 This is our first *paradigm-case of location*, which was worked out by considering the half-locative — half-mereological specification of the verb "to be": *to be* means — *to be in.*

It can be summarized as follows: I am in Schaan, if *each* part of me is there.

Thus, according to our preliminary analysis, we can say that

I *x is located in y* means: *each part of x is in y.*

8.4 Notice that location in general need not be transitive. For example, I am located in some academic institution which, in turn, is located in the network of European academies. But, clearly, *I* am not located in *this* network.

Therefore none of the structures discussed previously can serve as an *adequate* model of location. I.e., preorders, premereologies, *a fortiori* mereologies, as well as Leśniewskian ontologies cannot be used to model location in general. For all of them are transitive.

8.5 The above condition **I** clearly has some quasi-mereological connotations. But for reasons given above and those summarized in 2.7 location can be reduced neither to mereological inclusion — for mereologies are too strong (and therefore too restrictive), nor to preorders — for they are too formal.

As a matter of fact, exactly one component of the previous explication has a clear mereological connotation, namely the notion of "a part".

8.6 The schema **I** can thereby be rewritten semiformally as follows:

II *x is located in y iff* $\forall z\,(z \prec x \;\rightarrow\; z \textit{ is in } y)$
Here $\prec$ denotes a given part-relation.

Or, in a more general way:

III *x is located in* (*i.e. is in a suitable relation to*) *y iff*
$\forall z\,(z \prec x \;\rightarrow\; z \textit{ is related to } y)$
Here instead of "is in" — a particular case of the ontological primitive "is" — I am using its most general form "is related to".

8.7 In this way location is clearly connected with the part-whole relation. Indeed, for each particular type of part-whole relation $\prec$ and for a fixed variant of the primitive "is" we obtain the locative relation $L^{\prec}$:

IV $xL^{\prec}y$ *iff* $\forall z\ (z \prec x \ \rightarrow\ z$ ***is*** *related to* $y)$

This suggests, for given $\prec$ and E, the following definition of their *conjugate location*

V $xL^{\prec}y$ *iff* $\forall z\ (z \prec x \ \rightarrow\ zEy)$

We can simplify this further by reducing the number of primitives, i.e., by defining of the part-whole relation involved in terms of the basic ontological relation E after the clue of 5.1: $\prec = P_E$.

8.8 Three remarks are in order:

(A) Sometimes we are interested in *partial* rather than *total* location, i.e., in locating by means of putting in a given connection only *some* important or essential or relevant parts, *not all* of them. This generalizes our investigation in a rather obvious way, which because of lack of space I must skip here.

(B) As the most fundamental factor of location we picked out the *binary* relation "to be located in". But perhaps some locations depend upon further parameters as well?

For example, we can think of location as location *of something*, *somewhere, by somebody, in some way,* etc.

But still the binary-approach seems to be basic for any further, more sophisticated treatment. Therefore hereafter I shall concentrate on it.

(C) Location is not only important and basic notion in itself. It is also useful. Among other virtues, it enables us to define several important concepts. For example, *to localize* means: *to locate* something *in* some area; *to identify* means: *to locate* an object and to check that it indeed is the object we are looking for; *to combine* means: to *co-locate* and to *connect*; *to move* something means: to *dislocate* it, to change its location, etc.

8.9 Before going on to formalities I like to extend our intuition considering *a second paradigm case of location.*

Imagine the following system for the voice-opening of a door: There are four words which should be said in a given order. Four cards containing them, plus fifth, with sequence instruction, are put into a small suitcase which, in turn, is packed into a second, bigger suitcase. Both suitcases are, of course, closed.

Now, we can truly say that the Sesam-key-system S is located in the first suitcase S_1, which is in turn located with its content in the second suitcase S_2. Indeed, each of the five parts of S is in S_1, whereas each of the six parts of S_2 (i.e. the five parts of S plus S_1 itself) is in S_2:

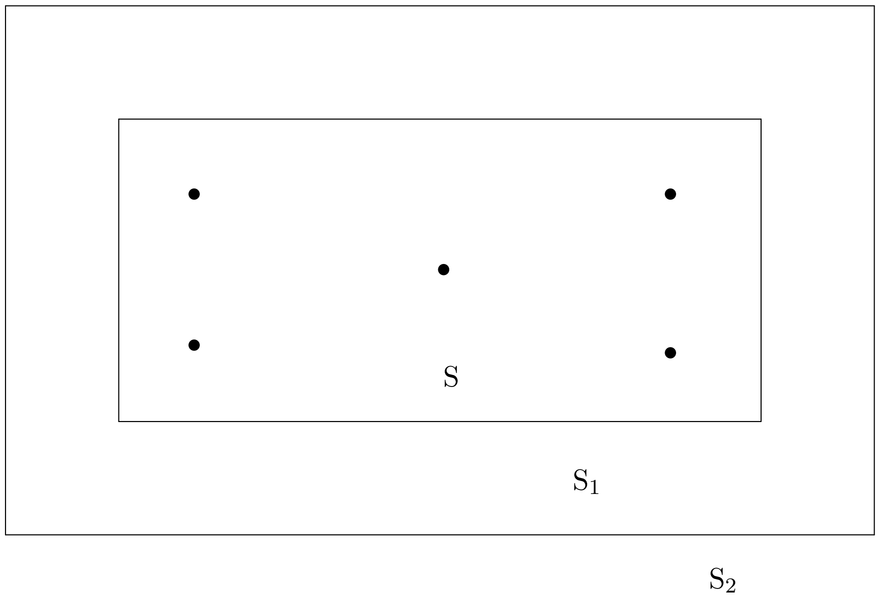
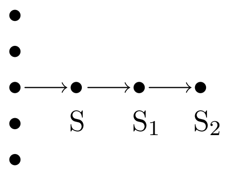


Fig. 7.

8.10 Observe that the above locative structure clearly has two aspects: an *internal* or *inside* one — described previously, and an *external* or *outside* one — easy to see when we like to find what is located in it by opening suitcases in their proper order: starting with S_2 through S_1 to the final reading (decoding) of S.

In our theory we therefore should distinguish two aspects or approaches to location: *internal*, which in fact was analysed in our first example, and *external*, which we just pointed out.

8.11 Notice that the situation described in Fig. 7 can be drawn in more schematic way:



or after further simplification:

$$\underset{S}{\bullet} \longrightarrow \underset{S_1}{\bullet} \longrightarrow \underset{S_2}{\bullet}$$

This is a typical order structure:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

As a matter of fact, several types of orders between numbers provide, as you will see, the most useful models of location.

8.12 As a matter of fact, our second paradigmatic example of location is a version of what is probably the most famous toy-model of location: *Russian dolls*, called also *Matrioshka*, a toy made of sequences of similar *babushkas* (Russian country-women) one inside another.

This famous toy can, in principle, be continued up infinity, visualizing in this way the basic order-structure of natural numbers:

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \cdots\cdots$$

which is a locative one (for an argument cf. §10.10).

8.13 To resume: Our examples teach us that:

i) Location has both an internal and an external aspect. The first is approached via the part-relation, the second one — by covering relation.

ii) From the internal point of view, to be located is to be located with all parts.

iii) From the external point of view, to be located is to be covered by any locating item's cover.

iv) Location can but need not be transitive.

v) We should be ready to discuss both finite and infinite cases of location.

Fundamental Conceps of Location

8.14 We proceed as follows: we start with a binary relation E used for "is related to". This is the only primitive notion of our theory. Other notions, including order-relations, are defined.

Recall that at the beginning of ch.5 all four such relations which are *a priori* possible were introduced, two of them — P and C - being preorders in general, two — H and D — being such for symmetric E only.

Clearly, using these four part-relations we can define (after the above receipt **V**) at least four conjugate location-relations. In the present study, I should like to concentrate my attention on two cases: P-location and C-location (with a few of their derivativess) - for these are general, natural and amusing.

Investigation of H- and D- location is postponed for the occasion when symmetric variants of "is" (like those concerning identity) will call for our attention.

8.15 P-location is *internally-oriented*, for the relation P goes from smaller to bigger, or from inside to outside. Call it *internal location* or simply *location*:

$$xLy := \forall z\ (zPx \to zEy), \qquad i.e.\ P^{-1}(x) \subseteq (y]$$

x is located in y iff any part of x is related to y.

8.16 The above definition works for *each* case of our primitive relation E.

Our intuition was fixed, however, by the case "to be in". In this paradigmatic case, the above definition can be read in a very natural way: *x is located in y iff each part of x is in y.*

By easy generalization location can, however, be defined in general, for each case of E, including all variants of "is".

Something similar will hold for C-location.

8.17 C-location is *externally-oriented*, for the cover-relation C goes from outside to inside (cf. §5.2). C is, in a sense, dual to P: x is covered by y, if any envelope of y envelops x as well. Thus C is externally oriented, checking if a bigger item contains a smaller one.

Call C -location *external location*, or simply *allocation*. Its definition is as follows:

$$xAy := \forall z\ (yCz \to xEz), \qquad or\ C(y) \subseteq [x)$$

x is allocated in y iff x is related to any cover of y.

8.18 P-location and C-location differ in quite remarkable way.

1° Let's check, first of all, that they indeed differ. To see this consider the following

Ex. 1 Let $U = \{x, y\}$, $E = \{xx, xy\}$.

Hereafter, we accept the convention that the concatenation xy of two objects means their ordered pair $\langle x, y\rangle$ and that in the picture of a given relation reflexive (irreflexive) points are depicted respectively by empty (nonempty) dots: $\circ$ and $\bullet$.

Thus the picture of $\langle U, E\rangle$ is as follows:

$$\underset{x}{\circ} \longrightarrow \underset{y}{\bullet}$$

Now we can calculate that $P = \{xx, yy, xy, yx\}$, i.e. that the part-relation P is full, whereas $C = \{xx, yy, xy\}$ which is not full. We should check each of eight cases involved. Consider two of them.

To see that yPx we must check that for any z, if zEy then zEx. Only the case $z := x$ is relevant. But xEx, as required.

On the other hand, $yx \notin C$, i.e. $\neg(yCx)$. Otherwise, for any z, if xEz then yEz. Take $z := x$. Then yEx, for xEx. This, however, is not true.

Hence $P \neq C$. By a similar calculation $L = \emptyset$, whereas $A = \{xx, xy\} = E$. Therefore $L \neq A$.

2° As we saw, *sometimes* location and allocation differ *extensionally*. In *each* case, however, they differ *intensionally*, stressing that we are dealing here with two opposite aspects of location.

8.19 This is stated implicitly in their definitions. For explication take a suitable example.

Consider the Russian doll in its recent version the Gorbachov's Matrioshka. This is built up in such a way that a figure of each Soviet general secretary (gensec) is contained in a figure of his successor: Lenin is in Stalin, Stalin in Malenkov, Malenkov in Khrutschov, Khrutschov in Brezhnev, Brezhnev in Andropov, Andropov in Chernenko and finally, God be praised, Chernenko in Gorbachov.

$$\underset{\mathbf{L}}{\bullet} \longrightarrow \underset{\mathbf{S}}{\bullet} \longrightarrow \underset{\mathbf{M}}{\bullet} \longrightarrow \underset{\mathbf{K}}{\bullet} \longrightarrow \underset{\mathbf{B}}{\bullet} \longrightarrow \underset{\mathbf{A}}{\bullet} \longrightarrow \underset{\mathbf{C}}{\bullet} \longrightarrow \underset{\mathbf{G}}{\bullet}$$

To be more formal, let $\Gamma = \{\mathbf{L}, \mathbf{S}, \mathbf{M}, \mathbf{K}, \mathbf{B}, \mathbf{A}, \mathbf{C}, \mathbf{G}\}$. Now, $E = \{\mathbf{LS}, \mathbf{SM}, \mathbf{MK}, \mathbf{KB}, \mathbf{BA}, \mathbf{AC}, \mathbf{CG}\}$.

It is easy to check that $P = \Delta_\Gamma \cup \{\mathbf{L}x : x \in \Gamma\}$, $C = \Delta_\Gamma \cup \{x\mathbf{G} : x \in \mathbf{G}\}$. Thus each gensec is both its own part and cover and, in addition, all gensecs have a common part: Lenin, and a common cover: Gorbachov.

To check location we should go in the "right-hand" direction, with the flow of time, from the toy's inside to its outside.

Allocation is investigated in the reverse direction, past-oriented, from the toy's outside figure — Gorbachov, by opening it successively until we reach its element — Lenin.

Using our definitions we can check that toy — location and allocation differ remarkably: $L = \{\mathbf{LS}\}$, $A = \{\mathbf{CG}\}$. Stalin is distinguished by location, for it is the only gensec in which something is located, namely Lenin. On the other hand, Gorbachov is the only gensec allocating something — namely Chernenko[22].

8.20 The third kind of location, *proper location*, is defined by combination of the two previous ones:

$$xPLy := xLy \wedge xAy$$

x is properly located in y iff it is both located and allocated in it.

8.21 To resume: Each binary relation generates its conjugate locative relations. We like to study them.

To this end, we use two part-relations: *internal* (P) and *external* (C). By means of them three locative relations have been defined: L, A and PL.

In this way we obtain a net of six connected relations; E, P, C, L, A, PL. Their properties and interconnections form a rich and well-motivated field of formal study — the relational theory of location, or simply — locative ontology.

Preliminary Observations

8.22 I am going to present several preliminary observations. I will be rather detailed as to L, results can be repeated for A.

Remember that definitions are also axioms. Up to now, *three* locative axioms were introduced, namely the definitions of L, A and PL.

[22] The question, is it a reasonable explanation of murdering strength of Stalin and relative disbelief of Gorbachov, is left to an ontologically oriented historiosopher.

Our preliminaries concern: first, what can be deduced from the definitions alone, and next, what can be deduced from definitions in one of the traditional realms outlined in chapters 5–7.

8.23 First of all, note that both location and allocation enjoy the following *condition of well-location*: what is located (allocated) is located with all its parts (allocated in all locating item's covers).

(43) $xLy \leftrightarrow \forall z\ (zPx \to zLy)$, *and* $xAy \leftrightarrow \forall z\ (yCz \to xAz)$

Proof. As regards location consider first the left-hand implication. I.e. let us suppose $\forall z\ (zPx \to zLy)$ and put $z := x$. In this way we obtain $xPx \to xLy$. But xPx, since P is reflexive; hence xLy.

For the converse implication assume xLy, i.e. $\forall z\ (zPx \to zEy)$, and that zPx. We need to prove that zLy, i.e., $\forall u\ (uPz \to uEy)$. Suppose additionally uPz. Using this and the second assumption we obtain uPx. Applying now the first assumption we obtain uEy, as required.

The argument in the case of allocation is analogous.

8.24 The above result deprives interest in iterating locative relations of both kinds in a simple ("natural") way. Indeed, if, for any $n \geqslant 0$, we put:

xL^0y	$:= xEy$	xA^0y	$:= xEy$
$xL^{n+1}y$	$:= \forall z\ (zPx \to zL^ny)$	$xA^{n+1}y$	$:= \forall z\ (yCz \to xA^nz)$

then, by (43), we immediately obtain

(44) *For any* $n \geqslant 1$ $xL^ny \leftrightarrow xLy$ *and* $xA^ny \leftrightarrow xAy$, *i.e.,* $L^n = L$ *and* $A^n = A$.

Therefore, the only remaining cases of natural equations are: $L = E$ and $A = E$.

Are these equations true in general? No, they hold only in the locative structures which we are going to describe in the next chapter.

8.25 Location, allocation and proper location are logically stronger, i.e. more narrow in scope, than is the original relation E:

(45) $xLy \to xEy$, $xAy \to xEy$ *and* $xPLy \to xEy$, *i.e.,* $L \leqslant E$, $A \leqslant E$ *and* $PL \leqslant E$.

Proof. For the first implication, recall that xLy means: $\forall z\ (zPx \rightarrow zEy)$. Take $z := x$. By reflexivity of P and by detachment we obtain the conclusion.

In the case of A we reason analogously. Next, the case of PL immediately follows.

8.26 We are now going to study the effect of imposing several well-known regularity conditions on our primitive relation E.

We start with the transitivity condition.

(46) *If E is transitive, then L and A are transitive also.*

Proof. For given x, y and z assume xLy and yLz. We need to prove that xLz, i.e., $\forall u\ (uPx \rightarrow uEz)$.

Enumerating our assumptions more carefully: we suppose $\forall u\ (uPx \rightarrow uEy)$, $\forall u\ (uPy \rightarrow uEz)$ and uPx. The first and the third assumption imply uEy. But E is transitive, hence by (4), $E \leqslant P$. Therefore uPy, which together with the second assumption entails the conclusion we need: uEz.

The argument for A is analogous.

The above implications are not reversible. To see this cf. Gorbachov's Matrioshka model of 8.19.

8.27 From (45) and (4) we immediately obtain

(47) *If E is transitive, then the following implications hold:*
$xLy \rightarrow xPy$, $xLy \rightarrow xCy$, $xAy \rightarrow xCy$ and $xAy \rightarrow xPy$.

Notice that *none of the above implications is reversible*, which follows by the two-element model presented in the Ex. 1 od 8.18.

Also *none of these implications entail transitivity.* To check such a claim we need, as usual, models. For the first and the third implication, i.e. for $L \leqslant P$ and $A \leqslant C$ see Gorbachov's matrioshka model, which satisfies both inequalities and is intransitive.

For the second implication, i.e. for $L \leqslant C$, consider the following

Ex. 2 $U := \{x, y, z\}$, $E := \{xx, xy, yz\}$, i.e.

$$\underset{x}{\circ} \longrightarrow \underset{y}{\bullet} \longrightarrow \underset{z}{\bullet}$$

We can easily calculate that $P = \Delta \cup \{xy, yx\}$, $C = \Delta \cup \{xz, yz\}$, whereas $L = \emptyset$ and $A = \{yz\}$. Thus $L \leqslant C$ and E is intransitive, as required.

The remaining case can be settled by similar modelling.

8.28 Joining (4), (45) and (47) together we can write in short:

(48) *For transitive* E, $L \cup A \leqslant E \leqslant P \cap C$.

8.29 Turn now to the case of reflexive E. First notice that

(49) *i)* xLx *iff* $\forall z\,(zPx \to zEx)$, *i.e.* $P^{-1}(x) \subseteq (x]$
ii) xAx *iff* $\forall z\,(xCz \to xEz)$, *i.e.* $C(x) \subseteq [x)$.

Hence, applying (4) we obtain

(50) *i)* L *is reflexive iff* E *is reflexive*
ii) A *is reflexive iff* E *is reflexive.*

Combining (46) with (50) we obtain also

(51) *If* E *is a preorder, then both* L *and* A *are preorders.*

8.30 For reflexive E, using once again (4) and (45), we immediately obtain

(52) $L \cup A \cup P \cup C \leqslant E$.

Notice that the five relations I have mentioned occur in two triples, for in general:

(53) $yCx \wedge xAy \to xEx$ *and* $yPx \wedge xLy \to yEy$.

Therefore

(54) *For irreflexive* E*:* C^{-1} *and* P^{-1} *are disjoint respectively with* A *and* L*:* $C^{-1} \cap A = \emptyset$, $P^{-1} \cap L = \emptyset$.

Transitivity Laws

8.31 Turn now to transitivity laws, like those which have been listed for overlapping O in §6.11.

These describe connections made by superposition of two (or more) appropriate relations. If at least one of the relations involved introduces "smaller-bigger orientation", then transitivity laws are called, according to this orientation, either *monotonicity* or *antimonotonicity* laws.

8.32 Frequently transitivity laws are the crux of important claims which sometimes seem to have little in common with the question of transitivity.

For example, it is easy to see that our first preliminary observation (43) claims, in fact, the following monotonicity for, respectively, P and L, and C and A:

(55) $zPx \wedge xLy \rightarrow zLy,$ *and* $xAy \wedge yCz \rightarrow xAz.$

The first claim is the *law of left-monotonicity for location*: Any part of what is located is located in the same item, i.e., location is location with parts; whereas the second claim is the *law of right-hand monotonicity for allocation*: allocation means allocation in allocating item's covers.

8.33 Notice that the two accompanying laws: the right-monotonicity law for location and the left-monotonicity law for allocation hold in general as well:

(56) $xLy \wedge yPz \rightarrow xLz$ *and* $xCy \wedge yAz \rightarrow xAz.$

8.34 Hence, using also (47) and (55) we obtain

(57) $zPx \wedge xLy \rightarrow zEy,$ $xAy \wedge yCz \rightarrow xEz,$ $xLy \wedge yPz \rightarrow xEz$ *and* $xCy \wedge yAz \rightarrow xEz.$

8.35 By (45) we know that in general $L \leqslant E$ and $A \leqslant E$. But P and C need not be comparable with E. They can be bigger, smaller or even cross with the relation E.

However, by (55)–(57), we obtain that superposition of P and C respectively with L and C are in *each* case smaller than E.

(58) $P \circ L \leqslant L \leqslant E,$ $L \circ P \leqslant L \leqslant E,$ $A \circ C \leqslant A \leqslant E$ *and* $C \circ A \leqslant A \leqslant E.$

8.36 But P and C are reflexive. Hence for any binary relation R defined on their universe U: $R \leqslant R \circ P$, $R \leqslant P \circ R$, $R \leqslant R \circ C$ and $R \leqslant C \circ R$.

Therefore, (58) entails

(59) $P \circ L = L = L \circ P$ *and* $C \circ A = A = A \circ C.$

8.37 Consider now the problem of left and right monotonicity for P and C with respect to the original relation E.

Two of the four a priori given statements hold in general: the right-monotonicity for P and the left-monotonicity for C . Indeed

(60) $xEy \wedge yPz \rightarrow xEz$ *and* $xCy \wedge yEz \rightarrow xEz$;
If x *is in* y *and* y *is a part of* z *then* x *is in* y*, and if* x *is covered by an object which is in* z *then* x *is in* z*.*

The remaining two laws hold under a special proviso, for example, for preorders:

(61) *If* E *is a preorder, then* $xPy \wedge yEz \rightarrow xEz$ *and* $xEy \wedge yCz \rightarrow xEz$*. I.e., if* x *is a part of something which is in* z *then* x *is in* z *as well, and if* x *is in something which is covered by* z *then* x *is in* z*.*

8.38 Arguing in the same way as before we can justify an algebraic version of the last two claims:

(62) *In general:* $E \circ P = E$ *and* $C \circ E = E$*, whereas under the proviso that* E *is a preorder relation:* $P \circ E = E$ *and* $E \circ C = E$*.*

8.39 Passing to monotonicity laws for two locative-relations which interest us especially, I will list two families of laws: the first for transitive E, the second for preorders

(63) *Let* E *be transitive. Then* $xEy \wedge yLz \rightarrow xEz$*,* $xEy \wedge yAz \rightarrow xEz$ *and* $xLy \wedge yEz \rightarrow xLz$. Informal reading: *If* x *is in something which is located in* z *then* x *is in* z*, if* x *is in something which is allocated in* z *then* x *is in* z*, and whatever is located in a case of* z *is located in* z *itself.*

Proof. All cases follow immediately from the definitions.

Consider, for example, the last implication. To this end assume that yEz and xLy, i.e., for any w, $wPx \rightarrow wEy$. In addition we assume the supposition of xLz, i.e., that for a given u, uPx. By particularization of the second assumption we obtain: uEy. This, by the first assumption and transitivity of E implies that uEz. Therefore xLz, as required.

(64) *If* E *is a preorder relation, then* $xEy \wedge yLz \rightarrow xLz$*,* $xEy \wedge yAz \rightarrow xAz$ *and* $xAy \wedge yEz \rightarrow xAz$. Speaking informally: *If* x *is in something which is located (allocated) in* z*, then* x *is in* z*; and whatever is allocated in a case of* z *is allocated in* z *itself.*

The proof is similar to the last one.

Finally, observe that *none of the above six laws is true in general.* As a matter of fact, they are weaker than their assumptions.

8.40 Expressing the above laws in terms of superposition we have

(65) *i) If* E *is transitive, then* $E \circ L \leqslant E$, $E \circ A \leqslant E$ *and* $L \circ E \leqslant L$
ii) If E *is a preorder, then* $E \circ L \leqslant L$, $E \circ A \leqslant A$ *and* $A \circ E \leqslant A$.

8.41 To conclude, transitivity laws are indeed the most natural laws comparing the 5 relations under investigation: E, P, C, L and A. A priori there is quite a lot[23] of such laws, 19 of which we have considered explicitly: the classical transitivity law for E — in chapters 4 and 5, and 18 laws discussed in this subchapter. Because of limitations of space, an investigation of the further cases is left for another occasion.

9. LOCATIVE ONTOLOGIES

Axioms

9.1 Now we are passing to a rather subtle question of axiomatization of locative structures. Which axioms should be added to obtain a more adequate formalisation of the relational concept of location?

This is both a formal and an essential question. We need axioms both well-motivated and rich in consequences. For axioms are judged by their fruits.

9.2 First observe that we can accept as axiom any well-motivated law of locative transitivity which is not generally valid.

As we just learned, there are plenty of candidates. I will discuss them later on.

9.3 Consider now the following way of finding axioms:

Suppose we are interested in the question: How many? I.e., for a given relational universe $\langle U, E \rangle$ and for its locative relations L and A, we would like to know how many items are located (allocated) one in another?

The *extreme* positions we can consider are as follows:

The strongest one, anything is located (allocated) in everything:

FUL $\forall x \forall y\, xLy$ — *full location,*
FUA $\forall x \forall y\, xAy$ — *full allocation.*

[23] Calculation of the number of such laws depends on further assumptions, cf. forthcoming remarks concerning relational syllogistic.

It is easy to see that these axioms are so strong that they axiomatize full structures:

(66) **FUL** *or* **FUA** *holds in* $\langle U, E\rangle$ *iff* E *is full, i.e.,* $\forall x \forall y\, xEy$.

Hence, both candidates, being too strong are uninteresting.

The opposite view that a relational structure has *no* case of location (allocation): $\neg\exists x \exists y\, xLy$ or $\neg\exists x \exists y\, xAy$ is also pathological, hence uninteresting.

9.4 The next assumptions of this type are assumptions of *nonemptiness of locative relations*:

NEL $\exists x \exists y\, xLy$
NEA $\exists x \exists y\, xAy$

These are more interesting conditions. Clearly, both **NEL** and **NEA** hold only in nonempty structures, i.e., they imply that $E \neq \emptyset$.

Query: Find suitable conditions to reverse the above implications. Characterize both **NEL** and **NEA**.

9.5 Turn to our fundamental question which generates the set of basic axioms concerning location.

Recall that we investigate interconnections between the following five relations: the original relation E which was introduced to formalize the verb "to be"; its two conjugate preorders P and C which, in the realm of E, formalize two basic part-relations, respectively "to be a part of" and "to be covered by"; and its two conjugate locative relations: internal — L and external — A.

The basic problem of the theory of location is to compare these five relations.

By now we know that in general $L \leqslant E$ and $A \leqslant E$ and we know cases in which $E \leqslant P$ and $E \leqslant C$ (for transitive E), $P \leqslant E$ and $C \leqslant E$ (for reflexive E), and $P = E = C$ (for preorders).

The remaining cases, however, are still obscure.

The main family of our axioms is introduced to answer those *questions of comparison*.

9.6 The basic question of this sort is: In which case is the starting relation E itself locative? In other words, characterize structures in which "to be related" equals "to be located in".

Previously two types of location were distinguished: internal (L) and external (A). Having this in mind we are ready to list the four a priori possible *positive* answers to our question.

The relational universe $\langle U, E\rangle$ is said to be:

Internally locative iff it satisfies the following axiom:

INL $$E = L,$$

i.e., if "to be related" is "to be located";

Externally locative iff it satisfies the following axiom:

EXL $$E = A,$$

i.e., if "to be related" is "to be allocated";

Locative iff it is both internally and externally located, i.e., it satisfies

L $$L = E = A$$

i.e., "to be related" is "to be both located and allocated";

Weakly locative iff it is either internally or externally locative or both, i.e., it satisfies:

WL $$E = L \vee E = A.$$

Notice that the convention A of §4.11 has introduced other names for the above axioms: **EL** for **INL**, **EA** for **EXL**, and **LEA** for **L**.

9.7 *Locative ontologies* of appropriate kinds are structures satisfying suitable axioms from the above list.

Observe that locative ontologies are axiomatized according to a general recipe used, with some success, several times before: For given relations consider cases where their graphs collapse, i.e., they are set-theoretically (or extensionally) equal.

9.8 The above four axioms constitute our *first*, fundamental set of axioms for location.

Two other sets are connected with comparison of L with P, A with C , and L with A.

9.9 It is easy to see that L and P as well as A and C are, in general, mutually incomparable.

Outside of the realm of preorders location means something other than "to be a part of".

To see this consider the following model:

Ex. 3 It contains two pieces which mirror each other, i.e., everything which is in the first is in the second and conversely. In such a case we can say that the first item is in the second and, conversely, the second is in the first.

In other words, our model constitutes an elementary loop:

$$x \bullet \rightleftarrows \bullet\ y$$

Here $U := \{x, y\}$ and $E := \{xy, yx\}$. Now it is easy to calculate that $P = \{xx, yy\} = C$, whereas $L = \{xy, yx\} = E = A$.

Hence our loop satisfies all axioms of the first group. Therefore, it is a locative structure.

But its part-relations are strongly incomparable with suitable locative relations, for they are mutually disjoint with them: $P \cap L = \emptyset$ and $C \cap A = \emptyset$.

Comparison of part- and locative-relations is therefore an essential question. It fully deserve to be decided by axioms.

9.10 By the way, notice that the above reasoning justifies the following useful observation:

(67) *If P and C are the smallest possible, i.e. if $P = \Delta = C$ then the structure $\langle U, E\rangle$ is locative, i.e., it fulfils $L = E = A$.*

For, under our assumption, $\forall z\ (zPx \to zEy)$ is equivalent to xEy, hence $xLy \leftrightarrow xEy$. Analogously for $A = E$.

Observe that instead of $P_E = \Delta$ we can assume $C_E = \Delta$, because

(68) $P_E = \Delta$ *iff* $C_E = \Delta$.

Cf. §9.34 below.

9.11 The second family of locative comparison axioms is as follows:

LP	$L \leqslant P$	***AC***	$A \leqslant C$
PL	$P \leqslant L$	***CA***	$C \leqslant A$
LP	$L = P$	**AC**	$A = C$

and for the case of strong incomparability:

LDSP $L \cap P = \emptyset$ ***ADSC*** $A \cap C = \emptyset$

The meaning of the axiom **LP** is clear: location and parthood relations coincide. To emphasize an implicit connection with preorders (cf. (114) in §10.18) the axiom **LP** is called the *axiom od prelocation*. Similarly, **AC** is the *axiom of preallocation*.

9.12 The third and final group of comparison axioms concerns comparison of the two locative relations, L with A.

By Gorbachov's Matrioshka model we know that in general L and A are incomparable, for there $L \cap A = \emptyset$ (cf. 8.19).

Hence any comparison is essential. The four axioms below exhaust all regular situations:

LA $L \leqslant A$
AL $A \leqslant L$
LA $L = A$
LDSA $L \cap A = \emptyset$

9.13 The main body of locative ontology is yielded by investigations of the consequences of several groups of the above axioms.

Immediate Consequences of the Axioms

9.14 Inquiry into consequences of locative axioms enumerated previously is a rather complex exercise. Therefore it needs some organization.

I will try to be rather scrupulous as to the proper axioms of location from the first group, being much more sketchy on the others.

The order of my discussion is as follows: I will start with immediate reformulations and consequences of the axioms under investigation, passing next to several indirect consequences. In subsequent parts of the paper I will try to clarify further the structure of the realm of locative ontologies. Finally, I will discuss several applications of locative ontology, including philosophical ones.

9.15 To throw light on the meaning of the basic locative axioms **INL**, **EXL** and **L** the following definitions are introduced:

For a given relation E on its universe U an object x is *well-locating* (*well-allocating*) iff all its subobjects are located (allocated) in it:

$$\begin{aligned} WL(x) &:= \forall y\,(yEx \rightarrow yLx) \\ WA(x) &:= \forall y\,(yEx \rightarrow yAx). \end{aligned}$$

For symmetry's sake, x is said to be *well-located (well-allocated)* iff any of its overobject locates (allocates) it:

$$\begin{aligned} LW(x) &:= \forall y\,(xEy \rightarrow xLy) \\ AW(x) &:= \forall y\,(xEy \rightarrow xAy). \end{aligned}$$

In quite a lot of cases both locating (allocating) and locative (allocative) items occur quite often. Also, in quite a lot of universes they play a quite noticable role. Therefore it is an useful exercise to characterize well-locating (allocating) and well-located (allocated) objects of a given relational structure.

Now we see the meaning of our axioms. The axiom **INL** picks spaces in which *each* object is both well-located and well-locating. Similarly, the axiom **EXL** distinguishes spaces in which *each* object is both well-allocated and well-allocating, whereas the axiom **L** says the same both for location and allocation.

9.16 Both axioms of internal and external location have quite a lot of illuminating equivalents.

Consider first **INL**. In its developed form it is

INL1 $$xEy \leftrightarrow \forall z\,(zPx \rightarrow zEy)$$

I.e., it exactly expresses the idea of location through E by means of P. To see this, recall that the relation E is the only primitive notion of the theory of location. P is defined by it in such a way that it really captures essential features of the relation "to be a part of". Hence the axiom **INL1** says: to be related is to be related with *all* parts, but this means: to be *fully* located.

9.17 In turn, by elimination of P we obtain the explicit form of **INL** which is an implicit "axiomatic" definition of the primitive E in the theory of internal location:

INL2 $$xEy \leftrightarrow \forall z\,[\forall u\,(uEz \rightarrow uEx) \rightarrow zEy]$$

Extracting quantifiers we obtain the prenex-form of the axiom

INL3 $\forall x \forall y \forall z \exists u\, [xEy \to ((uEz \to uEx) \to zEy)]$

Hence, in spite of its apparent clarity, the axiom of internal location has a rather strong logical form. It is a first order formula of the type $\forall^3\exists$, hence implicitely it is an *existential* formula.

9.18 In the light of (45) we immediately obtain

(69) **INL** *is equivalent to* ***EL****:* $E \leqslant L$.

Indeed, ***EL*** is expressed by

INL4 $xEy \to \forall z\, (zPx \to zEy)$

which, by (45), is equivalent to **INL**.

9.19 Moreover, the last formula is exactly the expression of the first monotonicity law stated in (61) for preorders. Hence

(70) **INL** *iff* $zPx \wedge xEy \to zEy$

which, in turn, enables us to obtain the following generalization of one of the facts stated in (62):

(71) **INL** *iff* $P \circ E = E$.

In this way facts which were previously stated for preorders can be generalized to, what will be proved, the broader domain of locative structures.

To conclude, some transitivity laws which are not generally true occur to be very natural axioms.

9.20 Observe that

(72) $E \leqslant P$ *iff* $E \leqslant C$*, and* $P \leqslant E$ *iff* $P \leqslant C$.

Indeed, both sides of the first equivalence are, by (4), equivalent to transitivity of E; whereas both sides of the second condition are equivalent to E reflexivity. Conditions compared in (72) must therefore be equivalent.

This, however, does not mean that P and C are in all cases comparable. This regularity is achieved only in the case of locative ontologies.

9.21 In fact, internally locative structures are exactly those which make P weaker than C:

(73) **INL** *iff* ***PC***: $P \leqslant C$.

Proof. By (69) we need instead of **INL** to consider only its right-hand implication: $xEy \to xLy$, which is equivalent to **INL4**: $xEy \to \forall z\ (zPx \to zEy)$, which in sequence is logically equivalent to an open formula $xEy \to (zPx \to zEy)$. This, in turn, is equivalent to $zPx \to (xEy \to zEy)$, which is again equivalent to $zPx \to \forall y\ (xEy \to zEy)$. This, finally, is equivalent to $zPx \to zCx$, i.e., $P \leqslant C$.

As regards the logic behind (73), the crux of the above proof lies in the free use of the exportation laws of classical quantifier logic, made possible by the very special distribution of variables in the starting formula.

9.22 Equivalents of **EXL** are, *mutatis mutandis*, similar to those for **INL**. Also, they can be proved by quite analogous arguments.

Therefore I can here only enumerate them, leaving proofs to the reader himself.

(74) **EXL** *is, in turn, equivalent to:*
EXL1 $xEy \leftrightarrow \forall z\ (yCz \to xEz)$,
EXL2 $xEy \leftrightarrow \forall z\ [\forall u\ (zEu \to yEu) \to xEz]$,
EXL3 $\forall x \forall y \forall z \exists u\ [xEy \to ((zEu \to yEu) \to xEz]$, *and*
EXL4 $xEy \to \forall z\ (yCz \to xEz)$.

Hence, the axiom **EXL** has the same logical complexity as **INL**. It is an $\forall^3\exists$-formula. Therefore, it is in fact an existential axiom.

9.23 Next, **EXL** is equivalent to the second monotonicity law from (61):

(75) **EXL** *iff* $xEy \wedge yCz \to xEz$

which, in turn, implies the following generalization of the second part of (62):

(76) **EXL** *iff* $E \circ C = E$.

On the other hand,

(77) **EXL** *is equivalent to* ***CP***: $C \leqslant P$.

9.24 The strong location axiom **L** is, by definition, the conjunction of **INL** and **EXL**. It says that $L = E = A$, which can be simplified as follows:

(78) **L** *iff* $E = L \cap A$.

Further equivalents of **L** can be obtained by appropriate conjunctions of equivalents of **INL** and **EXL**.

(79) *The following are equivalents of* **L**:
i) *Both* $zPx \wedge xEy \rightarrow zEy$ *and* $xEy \wedge yCz \rightarrow xEz$ *hold,*
ii) $P \circ E = E = E \circ C$
iii) **PC**: $P = C$.

Then we see that in the proper or strong locative structures both internal and external part-relation coincide. Their difference is, in a sense, the cause of the difference between two locative relations.

9.25 Clearly, equivalents of the weak axiom of location **WL** are disjunctions of appropriate conditions for internal and external location. Hence

(80) **WL** *iff* $E = L \cup A$.

9.26 The logical dependencies between locative axioms of the first group can be summarized in the following diagram:

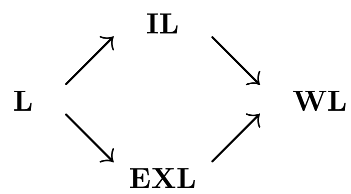


Fig. 8.

That the implications indicated by the arrows hold is clear. However, claims that *no arrow is reversible* and that **INL** *and* **EXL** *are independent* need suitable models. They will be provided in one of the subsequent subchapters, which is especially devoted to models.

9.27 Now we are going to discuss very briefly the second group of locative axioms, those listed in §9.11.

First of all, note that two of them are connected with reflexivity: ***PL***, as well as ***CA***, axiomatizes the well-known domain of reflexive orders:

(81) *i)* $P \leqslant L$ *iff* $P \leqslant E$ *iff* E *is reflexive*
ii) $C \leqslant A$ *iff* $C \leqslant E$ *iff* E *is reflexive.*

Proof. The second equivalence, both in i) and ii) has been stated previously (cf. §5.5, claim (4)).

To prove the left-hand implication of the first equivalence in i) assume $P \leqslant E$. We need to check that for any x and y, if xPy then xLy, i.e., that

$xPy \to \forall z\ (zPx \to zEy)$. Thus we can, in addition, assume xPy and zPx with the aim deducing zEy. From the second and the third assumption, by transitivity of P, we obtain zPy, which, by the first assumption, implies zEy, as required.

The right-hand implication is immediate, for in general $L \leqslant E$ (cf. (45)), hence $P \leqslant L$ does indeed imply $P \leqslant E$.

The remaining case of the first equivalence from ii) can be checked in a similar way.

9.28 The other two axioms, namely ***LP*** and ***AC***, are connected with transitivity, but not in so strict a way as the two previous ones.

(82) *If E is transitive, then $L \leqslant P$ and $A \leqslant C$ but not conversely: $L \leqslant P \wedge A \leqslant C$ does not imply the transitivity of E.*

Proof. The first claim immediately follows from (4) and (45). Indeed, if E is transitive then, by (4), $E \leqslant P$. On the other hand, by (45), $L \leqslant E$. Hence $L \leqslant P$. Similarly for ***AC***.

For the second claim recall Gorbachov's Matrioshka model, which satisfies both ***LP*** and ***AC*** but is intransitive.

9.29 In conclusion:

(83) *If E is a preorder, then $L = P$ and $A = C$.*

The above implication is not reversible. Indeed:

(84) *The strongest combination of axioms from the second group, i.e. the conjunction $L = P \wedge A = C$, does not imply that E is transitive.*

To see this consider the following

Ex. 4 Let $U = \{x, y, z\}$ and $E = \{xx, yy, zz, xy, yz\}$. I.e.

$$\begin{array}{ccccc}\circ & \longrightarrow & \circ & \longrightarrow & \circ \\ x & & y & & z\end{array}$$

It is easy to calculate that $P = \Delta \cup \{xy\} = L$ and $C = \Delta \cup \{yz\} = A$, but E is transparently non-transitive.

Combining this observation with the two previous ones we obtain that the axioms under consideration: $L = P$, $A = C$ and $L = P \wedge A = C$ are strictly intermediate between reflexivity and preordering. Indeed, each of them is weaker than reflexivity plus transitivity but stronger than reflexivity alone.

9.30 Observe that ***LP*** and ***AC*** are, like **INL** and **EXL**, semiexistential formulas. Indeed

(85) ***LP*** *is equivalent to:* $\forall x \forall y \exists z\ ((zPx \rightarrow zEy) \rightarrow xPy)$*; whereas*
AC *is equivalent to:* $\forall x \forall y \exists z\ ((yCz \rightarrow xEz) \rightarrow xCy)$.

9.31 Turn now to the remaining disjointness axioms **LDSP** and **ADSC**. We will pay attention only to the first axiom: $L \cap P = \emptyset$, all the observations stated below can automatically be extended to the second one.

First, observe that disjointness of L with P implies that L is irreflexive.

(86) $L \cap P = \emptyset \rightarrow L$ *is irreflexive.*

Indeed, by (2), $\Delta \subseteq P$. But $P \cap L = \emptyset$, hence $\Delta \cap L = \emptyset$. Applying now (1) we reach the desired conclusion.

Next observe that **LDSP** with E-transitivity implies that L is empty.

(87) **LDSP** *and* E *transitive* $\rightarrow L = \emptyset$.

Proof. Take a transitive relation E. By (4) and (45) we have: $L \leqslant E \leqslant P$. Hence $L \cap P = L$. But, by our first assumption, $L \cap P = \emptyset$. Therefore $L = \emptyset$, as required.

On the other hand, by (50) and (86) we obtain:

(88) $L \cap P = \emptyset$ *implies that* E *is not reflexive.*

Combining the last two statements together we have:

(89) *If* L *is nonempty and* **LDSP**, *then* E *is neither reflexive nor transitive.*

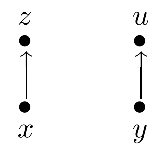
9.32 Finally consider the question of comparison of the two locative relations, L with A, which is settled by the axioms from the third group.

These axioms are clearly connected with the proper location axioms from the first group. Indeed

(90) **L** *implies* **LA**.

But not conversely. To see this, consider

Ex. 5 Let $U = \{x, y, z, u\}$ and $E = \{xz, yu\}$. I.e.



Here $L = \emptyset = A$, i.e **LA** holds, but **L** is not valid, as $L, A < E$.

On the other hand

(91) ***LA*** $\wedge$ **INL** $\rightarrow$ **EXL**, ***AL*** $\wedge$ **EXL** $\rightarrow$ **INL**.

A fortiori:

(92) *If L equals A, i.e.* **LA**, *then $\langle U, E\rangle$ is internally locative iff it is externally locative,i.e.,* **INL** $\leftrightarrow$ **EXL**.

The implications from (91) and (92) are also not reversible.

9.33 To finish our brief discussion of the last group of locative axioms, notice that they can be combined with the axioms of the second group to introduce further regularities, similar to those discussed in the previous section.

(93) ***PL*** $\wedge$ ***LA*** $\rightarrow$ ***PA***, ***LP*** $\wedge$ ***AL*** $\rightarrow$ ***AP***, ***LA*** $\wedge$ ***AC*** $\rightarrow$ ***LC*** *and* ***AL*** $\wedge$ ***CA*** $\rightarrow$ ***CL***.

Duality

9.34 We know that the four *derivative* relations under investigation: P, C, L and A in general differ both extensionally and intensionally. However, when taken in respective pairs: P with C and L with A, they are clearly connected. They, in a sense, mirror each other.

As a matter of fact, they are converse-dual. For a given relation E, its part-relation P_E can be considered as the converse of the covering relation taken for the relation E^{-1}. On the other hand, the E-covering C_E is the converse of the part relation for E^{-1}. Moreover, E-location equals the converse of E^{-1} allocation and *vice versa*. In symbols

(94) $P_E = (C_{E^{-1}})^{-1}, C_E = (P_{E^{-1}})^{-1}, L_E = (A_{E^{-1}})^{-1}$ and $A_E = (L_{E^{-1}})^{-1}$.

which immediately follows from

(95) $(P_E)^{-1} = C_{E^{-1}}$, $(C_E)^{-1} = P_{E^{-1}}$, $(L_E)^{-1} = A_{E^{-1}}$ and $(A_E)^{-1} = L_{E^{-1}}$.

Proof. Observe first that by the well known laws of the relation calculus: $(R^{-1})^{-1} = R$ and $R = S$ iff $R^{-1} = S^{-1}$, hence (94) indeed follows from (95).

To prove (95) consider first the case of P_E. Let $x(P_E)^{-1}y$, i.e., yP_Ex. By definition of P_E it is equivalent to $\forall z\ (zEy \rightarrow zEx)$ which, in turn, is equivalent to: $\forall z\ (yE^{-1}z \rightarrow xE^{-1}z)$, i.e. to $xC_{E^{-1}}y$.

Consider now the case of L_E. Proceeding similarly, $x(L_E)^{-1}y \leftrightarrow yL_Ex \leftrightarrow \forall z(zP_Ey \rightarrow zEx)$, which, by the previous case, is equivalent to: $\forall z\ (z(C_E)^{-1}y \rightarrow zEx)$ which, in turn, is equivalent to: $\forall z\ (yC_{E^{-1}}z \rightarrow xE^{-1}z)$, hence to: $xA_{E^{-1}}y$.

The remaining two cases can be checked analogously.

9.35 As immediate corollaries we have

(96) $L_E = P_E$ *iff* $A_{E^{-1}} = C_{E^{-1}}$, $A_E = C_E$ *iff* $L_{E^{-1}} = P_{E^{-1}}$, $L_E = E$ *iff* $A_{E^{-1}} = E^{-1}$, *and* $L_E = A_E$ *iff* $L_{E^{-1}} = A_{E^{-1}}$.

(97) *If E is symmetric, then **to be a part of** is the converse **to be covered by** and conversely. Moreover, **location** is the converse of **allocation** and vice versa:* $P = C^{-1}$, $C = P^{-1}$, $L = A^{-1}$ *and* $A = L^{-1}$.

Indeed, by (1), E is symmetric iff $E = E^{-1}$. Hence, under the proviso of the symmetry of E we can rewrite the equations in (94) in a form depending only on E. This is exactly what we need, for by our starting convention subscripts can be omitted, if unnecessary.

9.36 To conclude: By the duality of P with C and L with A we can reduce the number of notions of our theory. Instead of ***five***: a *primitive* — E and the four *derivatives* — P, C, L and A we can think of our subject as concerning interconnections between the original relation E and the *three* derivatives: converse-relation, part-relation and location with principles saying that internal location equals the converse of external location taken for the converse of the starting relation, etc.

However, in this way no essential reduction is in fact achieved, but only further clarification of the interconnections between parts and covers as well as between location and allocation.

Axioms Revisited

9.37 It is illuminating, in light of the previous discussion, to see our axioms once again.

The first group form the following axioms concerning location:

INL $L_E = E$
EXL $L_{E^{-1}} = E^{-1}$, hence
L $L_E = E$ and $L_{E^{-1}} = E^{-1}$.

In short, the proper axioms of location say respectively that the primitive relation E is locative, that its converse is locative, and that both the relation and its converse are locative.

The strongest axioms from the second group say respectively that

LP $L_E = P_E$
AC $L_{E^{-1}} = P_{E^{-1}}$.

I.e., that location as determined by the starting relation E equals to its parthood relation, and that location determined by the converse relation E^{-1} equals to its parthood relation.

Finally, the axiom **LA** is equivalent to: $(L_E)^{-1} = L_{E^{-1}}$. In plain words, the converse of location equals location determined by the converse.

9.38 To conclude: You can, if you like, replace the *two-aspect*, P — C and L — A, description of *a* given relational domain $\langle U, E\rangle$ by *one-aspect,* P — L, description of *two* domains: $\langle U, E\rangle$ and $\langle U, E^{-1}\rangle$.

Both approaches are indeed equivalent.

Preservation

9.39 Notice that the above observations, in particular (96), can be expressed also in the form of the following *preservation theorems*:

(98) *i)* $\langle U, E\rangle$ *is internally locative iff* $\langle U, E^{-1}\rangle$ *is externally locative, i.e., the relational frame verifies* **INL** *iff its converse verifies* **EXL**. *In symbols:* $\langle U, E\rangle \models$ **INL** *iff* $\langle U, E^{-1}\rangle \models$ **EXL**;

ii) $\langle U, E\rangle$ *is externally locating iff* $\langle U, E^{-1}\rangle$ *is internally locative, i.e.,* $\langle U, E\rangle \models$ **EXL** *iff* $\langle U, E^{-1}\rangle \models$ **INL**;

iii) $\langle U, E\rangle$ *is locative iff* $\langle U, E^{-1}\rangle$ *is locative, i.e.,* $\langle U, E\rangle \models$ **L** *iff* $\langle U, E^{-1}\rangle \models$ **L**.

In plain words: the taking of converses preserves proper location.

Similar statements hold also for the remaining two groups of axioms:

(99) *i)* $\langle U, E\rangle \models \boldsymbol{LP}$ *iff* $\langle U, E^{-1}\rangle \models \boldsymbol{AC}$
ii) $\langle U, E\rangle \models \boldsymbol{PL}$ *iff* $\langle U, E^{-1}\rangle \models \boldsymbol{CA}$
iii) $\langle U, E\rangle \models \mathbf{LP}$ *iff* $\langle U, E^{-1}\rangle \models \mathbf{AC}$

(100) *i)* $\langle U, E\rangle \models \boldsymbol{LA}$ *iff* $\langle U, E^{-1}\rangle \models \boldsymbol{AL}$
ii) $\langle U, E\rangle \models \boldsymbol{AL}$ *iff* $\langle U, E^{-1}\rangle \models \boldsymbol{LA}$
iii) $\langle U, E\rangle \models \mathbf{LA}$ *iff* $\langle U, E^{-1}\rangle \models \mathbf{LA}$